J. Fluid Mech. (2001), *vol.* 443, *pp.* 293–299. Printed in the United Kingdom

Gravity waves on shear flows

By JOHN MILES

Cecil H. and Ida M. Green Institute of Geophysics and Planetary Physics,
University of California, San Diego, La Jolla, CA 92093-0225, USA

(Received 5 August 2000 and in revised form 19 January 2001)

The eigenvalue problem for gravity waves on a shear flow of depth h and non-inflected velocity profile $U(y)$ (typically parabolic) is revisited, following Burns (1953) and Yih (1972). Complementary variational formulations that provide upper and lower bounds to the Froude number F as a function of the wave speed c and wavenumber k are constructed. These formulations are used to improve Burns's long-wave approximation and to determine Yih's critical wavenumber k_*, for which the wave is stationary ($c = 0$) and to which k must be inferior for the existence of an upstream running wave.

1. Introduction

Straight-crested, linear gravity waves of wavenumber $k > 0$ and wave speed c on the surface of a shear flow of ambient depth h and velocity $U(y)$ are governed by the Rayleigh equation

$$(U - c)(\phi'' - k^2\phi) - U''\phi = 0 \quad (0 < y < h, \quad ' \equiv \mathrm{d}/\mathrm{d}y) \tag{1.1}$$

and the bottom and free-surface boundary conditions

$$\phi = 0 \quad (y = 0), \quad (U - c)^2\phi' = g\phi \quad (y = h), \tag{1.2a,b}$$

where $\phi(y)\exp[ik(x - ct)]$ is a complex stream function. Following Burns (1953) and Yih (1972), I consider this eigenvalue problem for a velocity profile for which

$$U(0) = 0, \quad U(h) \equiv U_1 > 0, \quad U'(h) = 0, \quad U''(y) < 0. \tag{1.3a–d}$$

The simplest solution of (1.3) is the parabolic profile

$$U(y) = U_1 y(2h - y)/h^2, \tag{1.4}$$

which is realized for a nearly inviscid flow down a slightly inclined plane.

The basic problem is to determine the characteristic relation $f(\mathsf{c}, \mathsf{k}, F) = 0$ or, as proves more convenient, $G = G(\mathsf{c}, \mathsf{k})$, among the dimensionless parameters

$$\mathsf{c} = c/U_1, \quad \mathsf{k} = kh, \quad F = U_1/(gh)^{1/2}, \quad G = gh/U_1^2 \equiv 1/F^2. \tag{1.5a–d}$$

The still-water wave speed and drift speed are given by

$$\mathsf{C} \equiv C/U_1 = [(G/\mathsf{k})\tanh \mathsf{k}]^{1/2} \tag{1.6}$$

and

$$\mathsf{D} \equiv D/U_1 = \mathsf{c} \mp \mathsf{C} \begin{pmatrix} \mathsf{c} > 1 \\ \mathsf{c} < 0 \end{pmatrix} \tag{1.7}$$

for waves moving to the right/left (down/upstream). The dispersion relation $c = c(k)$ is implicitly determined by $G = G(c, k)$, and the corresponding group velocity is given by

$$c_g = \frac{\mathrm{d}}{\mathrm{d}k}[kc(k)] = c - k\left(\frac{\partial G/\partial k}{\partial G/\partial c}\right). \tag{1.8}$$

Burns (1953) solves (1.1)–(1.4) in the long-wave limit $k \downarrow 0$. Yih (1972) shows that the eigenvalue problem for prescribed k and F admits one solution with $c > 1$ for all $k > 0$ and a second solution with $c < 0$ if and only if $0 \leqslant k < k_*$, where k_* is a critical value of k for which the wave is stationary. There are no other solutions; accordingly, the singular point at $U = c$ lies outside the physical domain, and the admissible running waves are stable. The stationary ($c = 0$) wave, for which the singular point $U = 0$ lies on the lower boundary, is exceptional; however, the singular solution of the Rayleigh equation then may be excluded (see §4).

In the present investigation, I establish complementary variational formulations that provide upper and lower bounds to $G = G(c, k)$. As a first, brief example, I improve, and provide a measure of the truncation error in, Burns's long-wave ($k \ll 1$) approximation. As a more detailed example, I consider the stationary wave and derive variational approximations to the critical wavenumber k_* for the parabolic profile (1.4). These last results are relevant to the earlier controversy over the existence of upstream waves for large Froude numbers (see Benjamin 1962; Velthuizen & Wijngaarden 1969; Yih 1972; and Yih & Schultz 1999). In particular, the limit $F \uparrow \infty$ in (4.2) yields the asymptote

$$k_* h \sim (gh/\langle U^2 \rangle)^{1/2} \equiv 1/\langle F \rangle, \tag{1.9}$$

where $\langle F \rangle$ is the Froude number based on the r.m.s. flow speed $\langle U^2 \rangle^{1/2}$.

2. Variational formulations

Introducing the normalized streamline inclination θ and the dimensionless perturbation pressure $\tilde{\varpi}$ through the transformations (Miles 1962)

$$\phi(y)/U_1 h = (U - c)\theta(y) = (U - c)^{-1}\tilde{\varpi}'(y), \quad k^2 \tilde{\varpi}(y) = Q\theta'(y), \tag{2.1a,b}$$

where

$$y = y/h, \quad U(y) = U(y)/U_1, \quad Q = (U - c)^2, \tag{2.2a–c}$$

we transform (1.1) and (1.2a,b) to the complementary Sturm–Liouville systems

$$(Q\theta')' - k^2 Q\theta = 0 \quad (0 < y < 1, \quad ' \equiv \mathrm{d}/\mathrm{d}y), \tag{2.3}$$

$$(U - c)\theta = 0 \quad (y = 0), \quad Q\theta' = G\theta \quad (y = 1), \tag{2.4a,b}$$

and

$$(Q^{-1}\tilde{\varpi}')' - k^2 Q^{-1}\tilde{\varpi} = 0, \tag{2.5}$$

$$(U - c)^{-1}\tilde{\varpi}' = 0 \quad (y = 0), \quad G\omega' = k^2 Q\tilde{\varpi} \quad (y = 1), \tag{2.6a,b}$$

where c, k, and G are defined by (1.5), and either $c < 0$ or $c > 1$.

Multiplying (2.3) by θ, integrating by parts over $0 < y < 1$, invoking (2.4*a*, *b*), and dividing by $\theta_1^2 \equiv \theta^2(1)$, we obtain the variational integral

$$G = \frac{1}{\theta_1^2} \int_0^1 (\theta'^2 + k^2\theta^2) Q \, dy, \tag{2.7}$$

which is stationary with respect to variations of θ about the true solution of (2.3) and (2.4), is invariant under a scale transformation of θ (so that we may choose $\theta_1 = 1$), and provides an upper bound to the true value of G. Similarly,

$$\frac{1}{G} = \frac{1}{k^2\tilde{\varpi}_1^2} \int_0^1 \frac{(\tilde{\varpi}'^2 + k^2\tilde{\varpi}^2)}{Q} dy \tag{2.8}$$

provides a lower bound to the true value of G.

3. Long-wave approximation for running waves

Burns's (1953) solution of (2.3) and (2.4) for $k = 0$ is given by

$$\theta = \theta_1 \frac{R(y)}{R_1}, \quad R(y) = \int_0^y \frac{dy}{Q}, \quad R_1 \equiv R(1). \tag{3.1a–c}$$

Adopting (3.1*a*) as a trial function in (2.7), we obtain

$$G = G_0(c) + k^2 G_1(c), \quad G_0 = \frac{1}{R_1}, \quad G_1 = \frac{1}{R_1^2} \int_0^1 QR^2 \, dy. \tag{3.2a–c}$$

The error in (3.1*a*) is $O(k^2)$, whence that in the variational approximation (3.2*a*) is $O(k^4)$. We remark that (3.2) remains valid for $c \uparrow 0$, in which limit it reduces to the dominant term in (4.3).

Combining (1.6), (1.7) and (3.2*a*), we obtain

$$D = D_0(c) + k^2 D_1(c), \quad D_0 = c \mp G_0^{1/2}, \quad D_1 = \mp\tfrac{1}{2}(G_0^{-1/2} G_1 - \tfrac{1}{3} G_0^{1/2}) \begin{pmatrix} c > 1 \\ c < 0 \end{pmatrix}. \tag{3.3a–c}$$

It follows from (3.3*b*) and (3.2*b*) that

$$0 < D_0 < \langle U \rangle \quad \text{for} \quad 0 < -c < \infty \tag{3.4a}$$

and

$$1 > D_0 > \langle U \rangle \quad \text{for} \quad 1 < c < \infty, \tag{3.4b}$$

where $\langle U \rangle$ is the dimensionless, depth-averaged flow speed.

The results (3.2*b*, *c*) and (3.3*b*, *c*) are plotted in figures 1 and 2 for the parabolic profile (1.4), for which

$$U = 2y - y^2. \tag{3.5}$$

4. Stationary wave

The stationary wave ($c = 0$) is distinguished by the presence of the Rayleigh-equation singularity of exponents 0 and 1 at the lower boundary. The boundary condition (2.4*a*) then requires that the former solution be rejected, and hence that $\theta(y)$ be regular at $y = 0$.

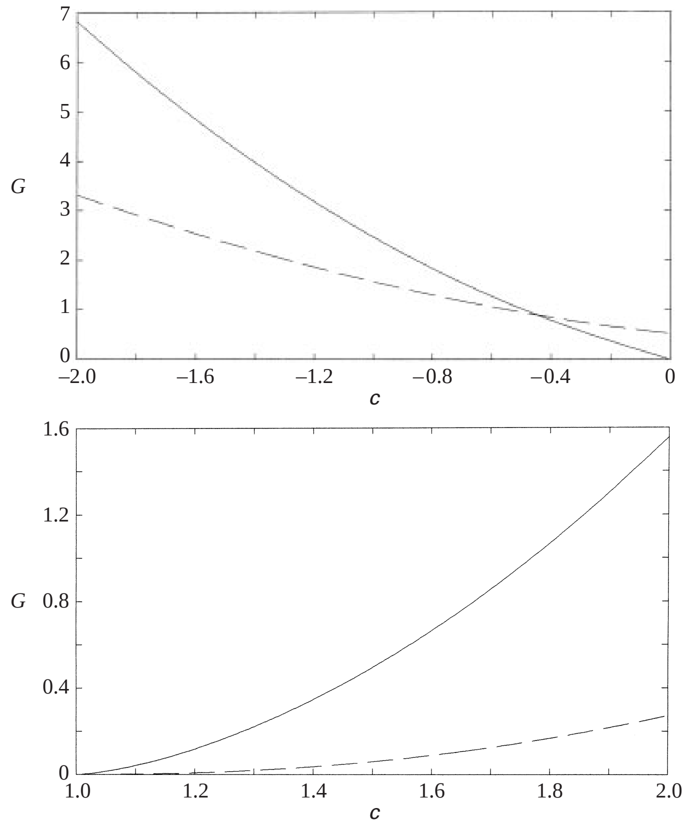

FIGURE 1. $G_0(c)$ (——) and $G_1(c)$ (– – –), as determined by (3.2*b*, *c*) for the parabolic profile (1.4).

Considering first the long-wave regime, we expand the solution of (2.3) and (2.4), with $Q = U^2$ therein, in powers of k^2 to obtain the trial function

$$\theta = 1 - k^2 \int_y^1 \frac{P}{U^2} \mathrm{d}y + O(k^4), \quad P \equiv \int_0^y U^2 \,\mathrm{d}y. \tag{4.1a,b}$$

Substituting (4.1) into (2.7) and integrating by parts, we obtain the upper bound

$$G = k^2 \left[P_1 - k^2 \int_0^1 (P/U)^2 \,\mathrm{d}y + k^4 \int_0^1 U^2 \left(\int_y^1 (P/U)^2 \,\mathrm{d}y \right)^2 \mathrm{d}y \right] + O(k^8), \tag{4.2}$$

in which $P_1 = \langle U^2 \rangle$ and the error is of the order of the square of that in the trial function. The limit $k \downarrow 0$ ($F \uparrow \infty$) of (4.2) yields (1.9).

For the parabolic profile (3.5), (4.2) reduces to

$$G = \tfrac{8}{15} k^2 - 0.06036 k^4 + 0.00194 k^6 + O(k^8), \tag{4.3}$$

the inversion of which yields (see figure 3)

$$k_*^2 = \tfrac{15}{8} G + 0.3180 G^2 + 0.1450 G^3 + O(G^4). \tag{4.4}$$

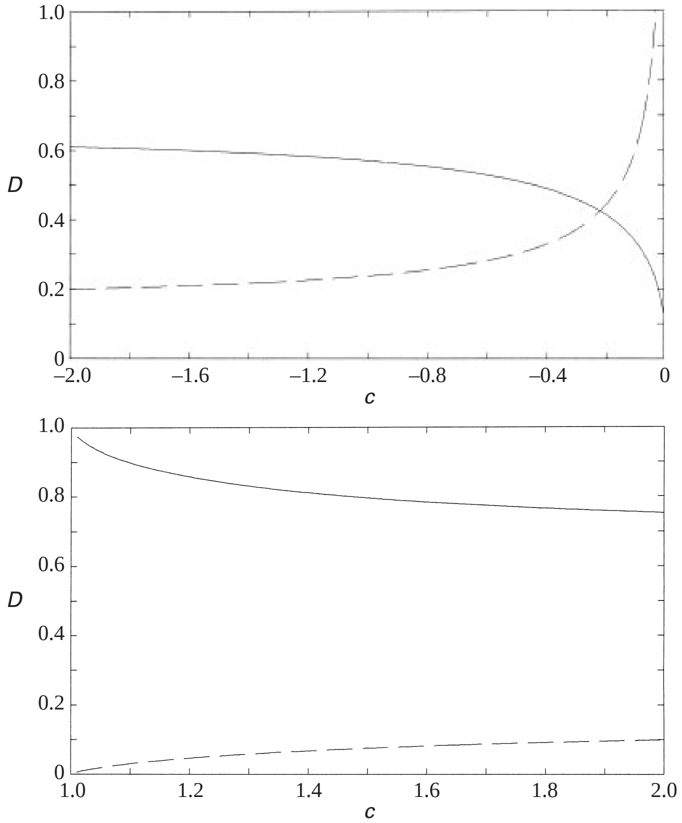

FIGURE 2. $D_0(c)$ (——) and $D_1(c)$ (– – –), as determined by (3.3*b, c*) for the parabolic profile (1.4).

The asymptotic solution of (1.1) and (1.2*a*) for $k \uparrow \infty$ (which is equivalent to that for uniform flow), $\phi \sim \sinh ky / \sinh k$, yields the short-wave trial function

$$\theta(y) = \frac{\sinh ky}{U(y)\sinh k}. \tag{4.5}$$

Substituting (4.5) into (2.7), integrating by parts, and invoking $U_1' = 0$, we obtain

$$G = \frac{1}{\sinh^2 k}\int_0^1 \left[k^2\cosh 2ky - k\frac{U'}{U}\sinh 2ky + \left(\frac{U'\sinh ky}{U}\right)^2\right]\mathrm{d}y \qquad (4.6a)$$

$$= k\coth k\,[1 - k^{-2} I(-U''/U)], \qquad (4.6b)$$

where

$$I[f(y)] = \frac{2k}{\sinh 2k}\int_0^1 f(y)\sinh^2 ky\,\mathrm{d}y \qquad (4.7a)$$

$$\sim \frac{1}{2}\sum_{n=0}^{\infty}(-)^n(2k)^{-n}[(\mathrm{d}/\mathrm{d}y)^n f(y)]_{y=1} \quad (k \uparrow \infty). \qquad (4.7b)$$

Turning to the complementary variational approximation, we substitute (4.5) into

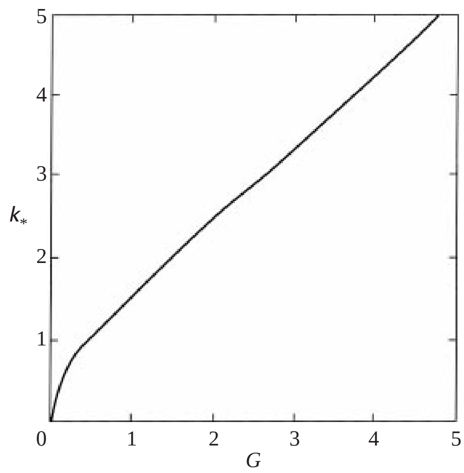

FIGURE 3. $k_*(G)$, as approximated by (4.4) for $G < 2.1$ and (4.12) for $G > 2.1$.

(2.1*b*) to obtain the trial function

$$\tilde{\omega} = \frac{U \cosh ky - k^{-1} U' \sinh ky}{\cosh k}. \tag{4.8}$$

Substituting (4.8) into (2.8) and proceeding as in (4.6), we obtain

$$\frac{1}{G} = \operatorname{sech}^2 k \int_0^1 \left[\cosh 2ky - \frac{U''}{U} \frac{\sinh^2 ky}{k^2} + \left(\frac{U''}{U}\right)^2 \frac{\sinh^2 ky}{k^4} \right] \mathrm{d}y \qquad (4.9a)$$

$$= k^{-1} \tanh k \{1 + k^{-2} I(-U''/U) + k^{-4} I[(U''/U)^2]\}. \qquad (4.9b)$$

For the parabolic profile (3.5), equations (4.6*b*), (4.7*b*) and the inverse of (4.9*b*) yield the lower and upper bounds (in each of which the first two terms are exact)

$$G = k - k^{-1} - \tfrac{1}{2}k^{-3} - k^{-5} + O(k^{-7}) \quad \text{and} \quad G \sim k - k^{-1} - \tfrac{3}{2}k^{-3} + k^{-5} + O(k^{-7}). \qquad (4.10a,b)$$

Empirical evidence (Miles 1962) suggests that the average of these bounds,

$$G = k - k^{-1} - k^{-3} + O(k^{-7}), \tag{4.11}$$

is superior to either of them. The inverse of (4.11)

$$k_* = G + G^{-1} + O(G^{-5}), \tag{4.12}$$

which intersects (4.4) at $G = 2.1$ and differs therefrom by less than 3% for $1.8 < G < 2.4$, is plotted in figure 3.

This work was supported in part by the Division of Ocean Sciences of the National Science Foundation Grant OCE98-03204, and by the Office of Naval Research Grant N00014-92-J-1171.

REFERENCES

Benjamin, T. B. 1962 The solitary wave in a stream with an arbitrary distribution of vorticity. *J. Fluid Mech.* **12**, 97–115.

Burns, J. C. 1953 Long waves in running water. *Proc. Camb. Phil. Soc.* **49**, 695–706.

Miles, J. W. 1962 A note on the inviscid Orr–Sommerfeld equation. *J. Fluid Mech.* **13**, 427–432.

Velthuizen, H. G. M. & Wijngaarden, L. van 1969 Gravity waves on a non-uniform flow. *J. Fluid Mech.* **39**, 817–829.

Yih, C.-S. 1972 Surface waves in flowing water. *J. Fluid Mech.* **51**, 209–220.

Yih, C.-S. & Schultz, W. W. 1999 Linear and nonlinear waves in flowing water. In *Fluid Dynamics at Interfaces* (ed. W. Shyy & R. Narayanan), pp 234–245. Cambridge University Press.